

Finite-size corrections in the $SU(2) \times SU(2)$ sector of type IIA string theory on $AdS_4 \times CP^3$

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Abstract

We consider finite-size corrections in the $SU(2) \times SU(2)$ sector of type IIA string theory on $AdS_4 \times CP^3$, which is the string dual of the recently constructed $\mathcal{N} = 6$ superconformal Chern-Simons theory of Aharony, Bergman, Jafferis and Maldacena (ABJM theory). The string states we consider are in the $\mathbb{R} \times S^2 \times S^2$ subspace of $AdS_4 \times CP^3$ with an angular momentum J on CP^3 being large. We compute the finite-size corrections using two different methods, one is to consider curvature corrections to the Penrose limit giving an expansion in $1/J$, the other by considering a low energy expansion in $\lambda' = \lambda/J^2$ of the string theory sigma-model, λ being the 't Hooft coupling of the dual ABJM theory. For both methods there are interesting issues to deal with. In the near-pp-wave method there is a $1/\sqrt{J}$ interaction term for which we use zeta-function regularization in order to compute the $1/J$ correction to the energy. For the low energy sigma-model expansion we have to take into account a non-trivial coupling to a non-dynamical transverse direction. We find agreement between the two methods. At order λ' and λ'^2 , for small λ' , our results are analogous to the ones for the $SU(2)$ sector in type IIB string theory on $AdS_5 \times S^5$. Instead at order λ'^3 there are interactions between the two two-spheres. We compare our results with the recently proposed all-loop Bethe ansatz of Gromov and Vieira and find agreement.

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1 Introduction and summary

Recently Aharony, Bergman, Jafferis and Maldacena proposed a new exact duality between gauge theory and string theory [1] based on earlier work on superconformal Chern-Simons theories [2].¹ The new duality is between a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory (ABJM theory), and type IIA string theory on $AdS_4 \times CP^3$. ABJM theory has $SU(N) \times SU(N)$ gauge symmetry with Chern-Simons like kinetic terms at level k and it is weakly coupled when the 't Hooft coupling $\lambda = N/k$ is small. Instead type IIA string theory on $AdS_4 \times CP^3$ is a good description when $1 \ll \lambda \ll k^4$.

Subsequently it was found in [4, 5] that the $SU(4)$ R-symmetry sector of ABJM theory is integrable at two-loop order. In particular one can consider the $SU(2) \times SU(2)$ sector of $SU(4)$. The operators in this sector are constructed from the single-trace operators of the form

$$\text{Tr}(A_{i_1} B_{j_1} A_{i_2} B_{j_2} \cdots A_{i_J} B_{j_J}) \quad (1)$$

with $A_{1,2}$ and $B_{1,2}$ transforming in the $(1/2, 0)$ and $(0, 1/2)$ of $SU(2) \times SU(2)$, respectively, and all scalars being in the bifundamental representation of $SU(N) \times SU(N)$. It was found in [4, 5] that this sector is described by two separate Heisenberg $XX_{1/2}$ spin chains, with $A_{1,2}$ corresponding to the up and down spins in the first Heisenberg chain, and $B_{1,2}$ to the second Heisenberg chain, the only interaction between them being the zero total momentum constraint of the magnons.

In [6] the $SU(2) \times SU(2)$ sector was studied from the string theory side. The $SU(2) \times SU(2)$ sector corresponds on the string theory side to considering an $\mathbb{R} \times S^2 \times S^2$ subspace of the $AdS_4 \times CP^3$ background. The $SU(2) \times SU(2)$ sector was approached by taking a low-energy sigma-model limit, with the result that at leading order the sigma-model action is that of two Landau-Lifshitz models added together. This is consistent with what one finds on the gauge theory side. Furthermore, a Penrose limit approaching the $SU(2) \times SU(2)$ sector was considered in [6] (see also [7, 5]) and a new Giant magnon solution was found in the $SU(2) \times SU(2)$ sector [6, 8]² (see also [5]). Combining these studies it was found that a magnon in the $SU(2) \times SU(2)$ sector has a dispersion relation that depends

¹For papers considering the ABJM theory see [3].

²For the giant magnon solution in $AdS_5 \times S^5$ see for example [9, 10, 11].

non-trivially on the coupling [5, 6]

$$\Delta = \sqrt{\frac{1}{4} + h(\lambda) \sin^2\left(\frac{p}{2}\right)}, \quad h(\lambda) = \begin{cases} 4\lambda^2 + \mathcal{O}(\lambda^4) & \text{for } \lambda \ll 1 \\ 2\lambda + \mathcal{O}(\sqrt{\lambda}) & \text{for } \lambda \gg 1 \end{cases} \quad (2)$$

where the weak coupling result is from [4, 5].

Very recently a proposal for an all-loop Bethe ansatz for the $\text{AdS}_4/\text{CFT}_3$ duality was put forward in [12]. This proposal combines the full $OSp(2, 2|6)$ superconformal symmetry with the results on integrability of ABJM theory found at weak coupling [4, 5], the interpolating dispersion relation (2) of [5, 6] and the study of integrability on the string theory side [13, 14, 15, 16]. The proposal utilizes many ingredients of the all-loop proposal for $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) [17, 18, 19].

In this paper we continue the study of integrability of the $\text{AdS}_4/\text{CFT}_3$ duality by computing the finite-size corrections to string states in the $SU(2) \times SU(2)$ sector of type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ with a large angular momentum J on \mathbb{CP}^3 . The string states are dual to single-trace operators of the form (1) in ABJM theory with $2J$ being the number of complex scalars in the operator.³ We compute the finite-size corrections using two different methods. The first method is to consider curvature corrections to the Penrose limit of [6] giving an expansion in $1/J$. The second method is to make a low energy expansion in $\lambda' \equiv \lambda/J^2$ of the string theory sigma-model, expanding around the $SU(2) \times SU(2)$ sigma-model limit of [6].

For the curvature corrections to the Penrose limit we follow the pioneering approach of [20, 21] in which curvature corrections to the BMN pp-wave [22] were considered for type IIB string theory on $\text{AdS}_5 \times S^5$. For simplicity we focus on string states in the $SU(2) \times SU(2)$ sector. We compute the $1/J$ correction to the energy of two different string states: $|s\rangle$ which is a two-oscillator state in the first $SU(2)$ and $|t\rangle$ which is a two-oscillator state with one oscillator in each of the $SU(2)$'s. The computation involves a new feature compared with that of [20, 21], namely that a $1/\sqrt{J}$ curvature correction appears in the Hamiltonian involving a transverse direction. This $1/\sqrt{J}$ correction appears as a second order correction at order $1/J$ giving a finite contribution to the energy after using zeta-function regularization.

For the state $|s\rangle$ we find the following energy

$$E_s = 2\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 1 + \frac{\lambda'}{J} \frac{4\pi^2 n^2}{\frac{1}{4} + 2\pi^2 n^2 \lambda'} \left(\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 2\pi^2 n^2 \lambda' \right) \quad (3)$$

where n is the oscillator number. For the state $|t\rangle$ we find

$$E_t = 2\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 1 + \frac{\lambda'}{J} \frac{4\pi^2 n^2}{\frac{1}{4} + 2\pi^2 n^2 \lambda'} \left(\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 2\pi^2 n^2 \lambda' - \frac{1}{2} \right) \quad (4)$$

Here $E = \Delta - J$. The computation of these energies is one of the main results of this paper.

Expanding the energies (3) and (4) of the two states $|s\rangle$ and $|t\rangle$ we find that at order λ' and λ'^2 the $1/J$ correction is what one would expect from knowing the $1/J$ correction to the $SU(2)$ sector of type IIB string theory on $\text{AdS}_5 \times S^5$. For the state $|t\rangle$ this entails that there is no interaction between the two $SU(2)$'s to this order which means that there are no $1/J$ corrections at order λ' and λ'^2 .

At order λ'^3 new interesting effects in the finite-size corrections appear. Most interestingly, the two $SU(2)$'s start to interact, and we get a non-zero $1/J$ correction to the string state $|t\rangle$. In particular,

³We pick the three Cartan generators of the $SU(4)$ R-symmetry (or the $SU(4)$ symmetry of \mathbb{CP}^3) R_1 , R_2 and R_3 such that $J = -R_3$, $S_z^{(1)} = (R_1 - R_2)/2$ and $S_z^{(2)} = (R_1 + R_2)/2$ where $S_z^{(1,2)}$ are the Cartan generators for the two $SU(2)$'s [6].

this means that the finite-size correction starts to deviate at this order from what one could naively expect from the $SU(2)$ sector in $\text{AdS}_5 \times S^5$.

Our second method to consider finite-size corrections consists in making a low-energy expansion of the sigma-model on $\text{AdS}_4 \times \mathbb{CP}^3$, with the energy $\Delta - J$ being small. This is an expansion in $\lambda' = \lambda/J^2$ around the $SU(2) \times SU(2)$ sigma-model limit of [6]. This method builds on the analogous low-energy sigma-model limit for the $SU(2)$ sector in $\text{AdS}_5 \times S^5$ [23, 24]. In parallel to the curvature correction, this computation also involves a new feature in comparison to [23, 24]. The new feature is that a field corresponding to a transverse direction has a non-trivial coupling to the fields of the $SU(2) \times SU(2)$ sector even though the field becomes non-dynamical in the $\lambda' \rightarrow 0$ limit.

To first order in λ' we have the result of [6] that the sigma-model is two Landau-Lifshitz models added together without any interaction terms. To second order in λ'^2 we find again no interaction terms and the sigma-model corresponds to two copies of the sigma-model found in the $SU(2)$ sector of $\text{AdS}_5 \times S^5$. At third order in λ'^3 new interesting effects appear and we get both interaction terms and new non-trivial terms for each of the $SU(2)$'s. We check for the two string states that our results are consistent with the results found from the curvature corrections to the Penrose limit by comparing with the energies (3) and (4) expanded up to third order in λ' .

Finally, we compare our results for the finite-size corrections to string states in the $SU(2) \times SU(2)$ sector to the newly proposed all loop Bethe ansatz [12]. We write down the explicit Bethe ansatz for the $SU(2) \times SU(2)$ sector that results from their proposal. Using this we compute the $1/J$ finite size corrections to the two string states up to order λ'^8 , for small λ' . Amazingly, we find perfect agreement up to that order. This constitutes a rather non-trivial check of the proposal of [12].

2 Preliminaries

ABJM theory is an $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $SU(N) \times SU(N)$ and level k . For $1 \ll \lambda \ll k^4$ it is well-described by type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ [1]. The $\text{AdS}_4 \times \mathbb{CP}^3$ background has the metric

$$ds^2 = \frac{R^2}{4} \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}_2^2 \right) + R^2 ds_{\mathbb{CP}^3}^2 \quad (5)$$

where the \mathbb{CP}^3 metric is

$$ds_{\mathbb{CP}^3}^2 = d\theta^2 + \frac{\cos^2 \theta}{4} d\Omega_2^2 + \frac{\sin^2 \theta}{4} d\Omega_2'^2 + 4 \cos^2 \theta \sin^2 \theta (d\delta + \omega)^2 \quad (6)$$

with

$$\omega = \frac{1}{4} \sin \theta_1 d\varphi_1 + \frac{1}{4} \sin \theta_2 d\varphi_2 \quad (7)$$

Here the curvature radius R is given by

$$R^4 = 32\pi^2 \lambda l_s^4 \quad (8)$$

Furthermore, the $\text{AdS}_4 \times \mathbb{CP}^3$ background has a constant dilaton with the string coupling given by

$$g_s = \left(\frac{32\pi^2 \lambda}{k^4} \right)^{\frac{1}{4}} \quad (9)$$

and it has a two-form and a four-form Ramond Ramond flux that will not be needed here, see for example [7, 6]. For our purposes it is convenient to make the coordinate change

$$\psi = 2\theta - \frac{\pi}{2} \quad (10)$$

such that the \mathbb{CP}^3 metric (6) takes the form

$$ds_{\mathbb{CP}^3}^2 = \frac{1}{4}d\psi^2 + \frac{1 - \sin\psi}{8}d\Omega_2^2 + \frac{1 + \sin\psi}{8}d\Omega_2'^2 + \cos^2\psi(d\delta + \omega)^2 \quad (11)$$

The $SU(2) \times SU(2)$ sector corresponds to the two two-spheres in the \mathbb{CP}^3 metric (11), parameterized as

$$d\Omega_2^2 = d\theta_1^2 + \cos^2\theta_1 d\varphi_1^2, \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2\theta_2 d\varphi_2^2 \quad (12)$$

On the string theory side, the $SU(2) \times SU(2)$ symmetry of the two two-spheres is a subgroup of the $SU(4)$ symmetry of \mathbb{CP}^3 . We can take the three independent Cartan generators for the $SU(4)$ symmetry to be

$$S_z^{(1)} = -i\partial_{\varphi_1}, \quad S_z^{(2)} = -i\partial_{\varphi_2}, \quad J = -\frac{i}{2}\partial_\delta \quad (13)$$

where $S_z^{(i)}$ are the Cartan generators of the two two-spheres.

On the gauge theory side, the $SU(2) \times SU(2)$ sector corresponds to consider single-trace operators of the form [4, 5]

$$\mathcal{O} = W_{i_1 i_2 \dots i_J}^{j_1 j_2 \dots j_J} \text{Tr}(A_{i_1} B_{j_1} \dots A_{i_J} B_{j_J}) \quad (14)$$

where $A_{1,2}$ and $B_{1,2}$ are the two pairs of complex scalars in ABJM theory, transforming in the $(1/2, 0)$ and $(0, 1/2)$ of $SU(2) \times SU(2)$, respectively, and all scalars being in the bifundamental representation of $SU(N) \times SU(N)$. Thus, on the gauge theory side $S_z^{(1)}$ counts the total spin for the $A_{1,2}$ scalars in (14) and $S_z^{(2)}$ for the $B_{1,2}$ scalars. Instead the bare scaling dimension of each scalar is $1/2$ which means that the total conformal dimension of (14) is $\Delta_0 = J$, Δ_0 being the bare scaling dimension. Indeed, one can define the $SU(2) \times SU(2)$ sector as consisting of the operators with $\Delta_0 = J$ [6].

The energy of the string states in units of the curvature radius R is dual to the scaling dimension Δ on the gauge theory side. In terms of the coordinates in the metric (5) we measure Δ as

$$\Delta = i\partial_t \quad (15)$$

3 Curvature corrections to Penrose limit

In this section we study curvature corrections to the Penrose limit of [6].

3.1 $SU(2) \times SU(2)$ Penrose limit of $\text{AdS}_4 \times \mathbb{CP}^3$

Consider the $\text{AdS}_4 \times \mathbb{CP}^3$ metric given by (5) and (11). We make the coordinate transformation

$$t' = t, \quad \chi = \delta - \frac{1}{2}t \quad (16)$$

This gives the following metric for $\text{AdS}_4 \times \mathbb{CP}^3$

$$ds^2 = -\frac{R^2}{4}dt'^2(\sin^2\psi + \sinh^2\rho) + \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2) \\ + R^2 \left[\frac{d\psi^2}{4} + \frac{1 - \sin\psi}{8}d\Omega_2^2 + \frac{1 + \sin\psi}{8}d\Omega_2'^2 + \cos^2\psi(dt' + d\chi + \omega)(d\chi + \omega) \right] \quad (17)$$

We have that

$$E \equiv \Delta - J = i\partial_{t'}, \quad 2J = -i\partial_\chi \quad (18)$$

Define the coordinates

$$v = R^2\chi, \quad x_1 = R\varphi_1, \quad y_1 = R\theta_1, \quad x_2 = R\varphi_2, \quad y_2 = R\theta_2, \quad u_4 = \frac{R}{2}\psi \quad (19)$$

We furthermore define u_1, u_2 and u_3 by the relations

$$\frac{R}{2} \sinh \rho = \frac{u}{1 - \frac{u^2}{R^2}}, \quad \frac{R^2}{4} (d\rho^2 + \sinh^2 \rho d\hat{\Omega}_2^2) = \frac{\sum_{i=1}^3 du_i^2}{(1 - \frac{u^2}{R^2})^2}, \quad u^2 = \sum_{i=1}^3 u_i^2 \quad (20)$$

Written explicitly, the metric (17) in these coordinates becomes

$$\begin{aligned} ds^2 = & -dt'^2 \left(\frac{R^2}{4} \sin^2 \frac{2u_4}{R} + \frac{u^2}{(1 - \frac{u^2}{R^2})^2} \right) + \frac{\sum_{i=1}^3 du_i^2}{(1 - \frac{u^2}{R^2})^2} + du_4^2 \\ & + \frac{1}{8} \left(\cos \frac{u_4}{R} - \sin \frac{u_4}{R} \right)^2 \left(dy_1^2 + \cos^2 \frac{y_1}{R} dx_1^2 \right) + \frac{1}{8} \left(\cos \frac{u_4}{R} + \sin \frac{u_4}{R} \right)^2 \left(dy_2^2 + \cos^2 \frac{y_2}{R} dx_2^2 \right) \\ & + R^2 \cos^2 \frac{2u_4}{R} \left[dt' + \frac{dv}{R^2} + \frac{1}{4} \left(\sin \frac{y_1}{R} \frac{dx_1}{R} + \sin \frac{y_2}{R} \frac{dx_2}{R} \right) \right] \left[\frac{dv}{R^2} + \frac{1}{4} \left(\sin \frac{y_1}{R} \frac{dx_1}{R} + \sin \frac{y_2}{R} \frac{dx_2}{R} \right) \right] \end{aligned} \quad (21)$$

a very convenient form to expand around $R \rightarrow \infty$.

The $SU(2) \times SU(2)$ Penrose limit $R \rightarrow \infty$ of [6] gives now the pp-wave metric⁴

$$ds^2 = dv dt' + \sum_{i=1}^4 (du_i^2 - u_i^2 dt'^2) + \frac{1}{8} \sum_{i=1}^2 (dx_i^2 + dy_i^2 + 2dt' y_i dx_i) \quad (22)$$

The light-cone coordinates in this metric are t' and v . We record here for completeness the two-form and four-form Ramond-Ramond fluxes

$$F_{(2)} = dt' du_4, \quad F_{(4)} = 3dt' du_1 du_2 du_3 \quad (23)$$

This is a pp-wave background with 24 supersymmetries first found in [26, 27]. See [7, 5] for other Penrose limits of the $\text{AdS}_4 \times \mathbb{CP}^3$ background giving the pp-wave background (22)-(23).

We see from (18) that

$$\frac{2J}{R^2} = -i\partial_v \quad (24)$$

Thus, the Penrose limit on the gauge theory side is the following limit

$$\lambda, J \rightarrow \infty \quad \text{with} \quad \lambda' \equiv \frac{\lambda}{J^2} \text{ fixed}, \quad \Delta - J \text{ fixed} \quad (25)$$

3.2 Bosonic string Hamiltonian

We now consider type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ in the above Penrose limit, including the curvature corrections in $1/R$. For simplicity we consider only the bosonic string modes. We set the string length $l_s = 1$ in the rest of this paper.

The bosonic string action is given by

$$I = \frac{1}{2\pi} \int d\tau d\sigma \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \quad (26)$$

Here $h^{\alpha\beta} = \sqrt{-\det \gamma} \gamma^{\alpha\beta}$ with $\gamma_{\alpha\beta}$ being the world-sheet metric. This means that $\det h = -1$, thus $h^{\alpha\beta}$ has only two independent components. The metric $G_{\mu\nu}$ is given by (21).

For convenience we define the momenta as

$$p_\mu = -h^{\tau\alpha} G_{\mu\nu} \partial_\alpha x^\nu \quad (27)$$

From this we see that

$$\dot{x}^\mu = -\frac{1}{h^{\tau\tau}} G^{\mu\nu} p_\nu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} x'^\mu \quad (28)$$

⁴See [25] for the analogous Penrose limit for the $SU(2)$ sector of $\text{AdS}_5 \times S^5$.

$$\mathcal{L} = -\frac{1}{2h^{\tau\tau}}G^{\mu\nu}p_\mu p_\nu + \frac{1}{2h^{\tau\tau}}G_{\mu\nu}x'^\mu x'^\nu \quad (29)$$

The Hamiltonian density is

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = -\frac{1}{2h^{\tau\tau}}(G^{\mu\nu}p_\mu p_\nu + G_{\mu\nu}x'^\mu x'^\nu) - \frac{h^{\tau\sigma}}{h^{\tau\tau}}x'^\mu p_\mu \quad (30)$$

Considering the two fields $\frac{1}{h^{\tau\tau}}$ and $\frac{h^{\tau\sigma}}{h^{\tau\tau}}$ as the two independent components of $h^{\alpha\beta}$, we can regard these two fields as Lagrange multipliers. This gives the constraints

$$G^{\mu\nu}p_\mu p_\nu + G_{\mu\nu}x'^\mu x'^\nu = 0, \quad x'^\mu p_\mu = 0 \quad (31)$$

We impose now the lightcone gauge

$$t' = c\tau, \quad p_v = \text{const.} \quad (32)$$

where c is a constant. The constant c can be fixed from the term $\frac{c}{2}\partial_\tau v$ in the full Lagrangian. In fact we have that $p_v = \partial\mathcal{L}/\partial\partial_\tau v$ which gives

$$c = \frac{4J}{R^2} = \frac{J}{\pi\sqrt{2\lambda}} \quad (33)$$

where we used that $\int_0^{2\pi} \frac{d\sigma}{2\pi} p_\chi = 2J$. Then the constraints (31) can be written as

$$G^{t't'}(p_{t'})^2 + G^{vv}(p_v)^2 + 2G^{t'v}p_{t'}p_v + 2G^{t'x_a}p_{t'}p_{x_a} + 2G^{vx_a}p_vp_{x_a} + G^{x_ax_b}p_{x_a}p_{x_b} + G^{y_ay_a}p_{y_a}p_{y_a} \\ + G^{u_iu_j}p_{u_i}p_{u_j} + G_{vv}(v')^2 + 2G_{vx_a}v'x'_a + G_{x_ax_b}x'_ax'_b + G_{y_ay_a}y'_ay'_a + G_{u_iu_j}u'_iu'_j = 0 \quad (34)$$

$$v'p_v + x'_ap_{x_a} + y'_ap_{y_a} + u'_ip_{u_i} = 0 \quad (35)$$

with $a, b = 1, 2$ and $i, j = 1, 2, 3, 4$. Eliminating v' in (34) using (35), one gets a quadratic equation for the light-cone Hamiltonian density $\mathcal{H}^{\text{lc}} = -p_{t'}$. Thus, we can solve the quadratic constraint (34) to obtain the lightcone Hamiltonian density, which we then expand up to $\mathcal{O}(\frac{1}{R^2})$

$$\mathcal{H}^{\text{lc}} = \mathcal{H}_{\text{free}}^{\text{lc}} + \mathcal{H}_{\text{int}}^{\text{lc}} \quad (36)$$

The complete expression for the Hamiltonian $\mathcal{H}_{\text{int}}^{\text{lc}}$ in terms of the momenta is however quite complicated even at the order $\mathcal{O}(\frac{1}{R^2})$. So we do not reproduce it here. It simplifies a lot instead when written in terms of the velocities at the zeroth order in the $\frac{1}{R}$ expansion.

To eliminate the momenta in terms of the velocities we should use eq.(27) with the leading order worldsheet metric $h^{\tau\tau} = -1$, $h^{\tau\sigma} = 0$. One gets

$$p_{x_1} = \frac{1}{8}(\dot{x}_1 + cy_1), \quad p_{x_2} = \frac{1}{8}(\dot{x}_2 + cy_2), \quad p_{y_1} = \frac{1}{8}\dot{y}_1, \quad p_{y_2} = \frac{1}{8}\dot{y}_2 \quad (37)$$

where by \dot{x}_a , \dot{y}_a we mean the velocities at the zeroth order in the $\frac{1}{R}$ expansion. The other momenta are standard. The leading term in the $\frac{1}{R}$ expansion gives the pp-wave quadratic Hamiltonian

$$\mathcal{H}_{\text{free}}^{\text{lc}} = \frac{1}{16c}[(x'_a)^2 + (y'_a)^2 + (\dot{x}_a)^2 + (\dot{y}_a^2)^2] + \frac{1}{2c} \sum_{i=1}^4 [(\dot{u}_i)^2 + (u'_i)^2 + c^2 u_i^2] \quad (38)$$

The interacting Hamiltonian contains two parts, one that goes like $1/R$ which is cubic in the fields and the other one that goes like $1/R^2$ which is quartic in the fields

$$\mathcal{H}_{\text{int}}^{\text{lc}} = \mathcal{H}_{\text{int}}^{(1)} + \mathcal{H}_{\text{int}}^{(2)} \quad (39)$$

where

$$\mathcal{H}_{\text{int}}^{(1)} = \frac{u_4}{8Rc} [(\dot{x}_1)^2 - (\dot{x}_2)^2 + (\dot{y}_1)^2 - (\dot{y}_2)^2 - (x'_1)^2 + (x'_2)^2 - (y'_1)^2 + (y'_2)^2] \quad (40)$$

and

$$\begin{aligned} \mathcal{H}_{\text{int}}^{(2)} &= \frac{1}{128R^2c^3} \left[4(\dot{x}_a x'_a + \dot{y}_a y'_a)^2 - \left((x'_a)^2 + (y'_a)^2 + (\dot{x}_a)^2 + (\dot{y}_a)^2 \right)^2 \right] \\ &+ \frac{1}{48R^2c} \left[3((\dot{x}_1)^2 - (x'_1)^2) y_1^2 + ((\dot{x}_2)^2 - (x'_2)^2) y_2^2 + c(\dot{x}_1 y_1^3 + \dot{x}_2 y_2^3) \right] + \dots \end{aligned} \quad (41)$$

the dots are for terms that are irrelevant in the computation of the spectrum of string states belonging to the $SU(2) \times SU(2)$ sector.

From the Hamiltonian densities one gets the Hamiltonian as

$$H_{\text{free}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{\text{free}}^{\text{lc}} d\sigma, \quad H_{\text{int}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{\text{int}}^{\text{lc}} d\sigma \quad (42)$$

The mode expansion for the bosonic fields can be written as

$$u_i(\tau, \sigma) = i \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\Omega_n}} \left[\hat{a}_n^i e^{-i(\Omega_n \tau - n\sigma)} - (\hat{a}_n^i)^\dagger e^{i(\Omega_n \tau - n\sigma)} \right] \quad (43)$$

$$z_a(\tau, \sigma) = 2\sqrt{2} e^{i\frac{\sigma\tau}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\omega_n}} \left[a_n^a e^{-i(\omega_n \tau - n\sigma)} - (\tilde{a}_n^a)^\dagger e^{i(\omega_n \tau - n\sigma)} \right] \quad (44)$$

where $\Omega_n = \sqrt{c^2 + n^2}$, $\omega_n = \sqrt{\frac{c^2}{4} + n^2}$ and we defined $z_a(\tau, \sigma) = x_a(\tau, \sigma) + iy_a(\tau, \sigma)$. The canonical commutation relations $[x_a(\tau, \sigma), p_{x_b}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma')$, $[y_a(\tau, \sigma), p_{y_b}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma')$ and $[u_i(\tau, \sigma), p_j(\tau, \sigma')] = i\delta_{ij}\delta(\sigma - \sigma')$ follow from

$$[a_m^a, (a_n^b)^\dagger] = \delta_{mn}\delta_{ab}, \quad [\tilde{a}_m^a, (\tilde{a}_n^b)^\dagger] = \delta_{mn}\delta_{ab}, \quad [\hat{a}_m^i, (\hat{a}_n^j)^\dagger] = \delta_{mn}\delta_{ij} \quad (45)$$

Employing (45) and (38) we obtain the bosonic free Hamiltonian as

$$cH_{\text{free}} = \sum_{i=1}^4 \sum_{n \in \mathbb{Z}} \sqrt{n^2 + c^2} \hat{N}_n^i + \sum_{a=1}^2 \sum_{n \in \mathbb{Z}} \left(\sqrt{\frac{c^2}{4} + n^2} - \frac{c}{2} \right) M_n^a + \sum_{a=1}^2 \sum_{n \in \mathbb{Z}} \left(\sqrt{\frac{c^2}{4} + n^2} + \frac{c}{2} \right) N_n^a \quad (46)$$

with the number operators $\hat{N}_n^i = (\hat{a}_n^i)^\dagger \hat{a}_n^i$, $M_n^a = (a_n^a)^\dagger a_n^a$ and $N_n^a = (\tilde{a}_n^a)^\dagger \tilde{a}_n^a$, and with the level-matching condition

$$\sum_{n \in \mathbb{Z}} n \left[\sum_{i=1}^4 \hat{N}_n^i + \sum_{a=1}^2 (M_n^a + N_n^a) \right] = 0 \quad (47)$$

Using (33) the spectrum (46) reads

$$H_{\text{free}} = \sum_{i=1}^4 \sum_{n \in \mathbb{Z}} \sqrt{1 + \frac{2\pi^2 \lambda}{J^2} n^2} \hat{N}_n^i + \sum_{a=1}^2 \sum_{n \in \mathbb{Z}} \left[\left(\sqrt{\frac{1}{4} + \frac{2\pi^2 \lambda}{J^2} n^2} - \frac{1}{2} \right) M_n^a + \left(\sqrt{\frac{1}{4} + \frac{2\pi^2 \lambda}{J^2} n^2} + \frac{1}{2} \right) N_n^a \right] \quad (48)$$

3.3 Perturbative analysis of the string energy spectrum

We shall now compute finite size corrections to the energies of two oscillator states of the form

$$|s\rangle = (a_n^1)^\dagger (a_{-n}^1)^\dagger |0\rangle \quad (49)$$

with both oscillators in just one of the two $SU(2)$'s of the $SU(2) \times SU(2)$ sector, and of the form

$$|t\rangle = (a_n^1)^\dagger (a_{-n}^2)^\dagger |0\rangle \quad (50)$$

with an oscillator in each of the two $SU(2)$'s of the $SU(2) \times SU(2)$ sector.

At the first order in perturbation theory the Hamiltonian (40) does not contribute to the energies of the states (49) and (50). Its mean value on these states vanishes, so that we shall only have corrections to the energies at the order $\mathcal{O}(\frac{1}{R^2})$. We will thus have two contributions to the energy corrections, one that comes from computing at the second perturbative order the contribution of the term (40) and one that arises from the first perturbative order just by taking the mean value of the Hamiltonian (41) on the states $|s\rangle$ and $|t\rangle$. For these states we have respectively

$$E_{s,t}^{(2)} = \langle s, t | H_{\text{int}}^{(2)} | s, t \rangle + \sum_{|i\rangle} \frac{|\langle i | H_{\text{int}}^{(1)} | s, t \rangle|^2}{E_{|s\rangle, |t\rangle}^{(0)} - E_{|i\rangle}^{(0)}} \quad (51)$$

where $|i\rangle$ is an intermediate state with zeroth order energy $E_{|i\rangle}^{(0)}$.

The relevant part of the Hamiltonian (40) contributing to the second term in (51) written in terms of oscillators reads

$$H_{\text{int}}^{(1)} = \frac{i}{Rc\sqrt{2}} \sum_{m,l,r} \frac{1}{\sqrt{\omega_m \omega_l \Omega_r}} \left[\left(\omega_m - \frac{c}{2} \right) \left(\omega_l - \frac{c}{2} \right) + ml \right] (\hat{a}_{-r}^4)^\dagger [(a_{-m}^2)^\dagger (a_l^2) - (a_{-m}^1)^\dagger (a_l^1)] \quad (52)$$

We have written operator monomials in normal ordered form. The normal ordering ambiguity that would arise in $H_{\text{int}}^{(1)}$ would not contribute to the matrix elements in (51).

The quartic part of the interaction Hamiltonian (41) which is relevant for computing the $\mathcal{O}(\frac{1}{R^2})$ corrections to the pp-wave spectrum reads

$$\begin{aligned} H_{\text{int}}^{(2)} = & \sum_{m,p,q,r} \frac{[(a_{-m}^1)^\dagger (a_{-p}^1)^\dagger a_q^1 a_r^1 + (a_{-m}^2)^\dagger (a_{-p}^2)^\dagger a_q^2 a_r^2] \delta(m+p+q+r)}{R^2 \sqrt{\omega_m \omega_p \omega_q \omega_r}} \\ & \left\{ \frac{-mp - qr + 4mq}{4c} + \frac{\omega_m + \omega_q - c}{4} - \frac{mpqr}{2c^3} \frac{1}{4c} \left[\left(\omega_m - \frac{c}{2} \right) \left(\omega_p - \frac{c}{2} \right) + \left(\omega_q - \frac{c}{2} \right) \left(\omega_r - \frac{c}{2} \right) \right. \right. \\ & + 4 \left(\omega_m - \frac{c}{2} \right) \left(\omega_q - \frac{c}{2} \right) \left. \right] + \frac{1}{2c^3} \left[mp \left(\omega_q - \frac{c}{2} \right) \left(\omega_r - \frac{c}{2} \right) + qr \left(\omega_m - \frac{c}{2} \right) \left(\omega_p - \frac{c}{2} \right) \right] \\ & - \frac{1}{2c^3} \left(\omega_m - \frac{c}{2} \right) \left(\omega_p - \frac{c}{2} \right) \left(\omega_q - \frac{c}{2} \right) \left(\omega_r - \frac{c}{2} \right) \left. \right\} \\ & - \sum_{m,p,q,r} \frac{(a_{-m}^1)^\dagger (a_p^1) (a_{-q}^2)^\dagger a_r^2 \delta(m+p+q+r)}{R^2 c^3 \sqrt{\omega_m \omega_p \omega_q \omega_r}} \left\{ \left[\left(\omega_m - \frac{c}{2} \right) \left(\omega_p - \frac{c}{2} \right) - mp \right] \times \right. \\ & \left. \left[\left(\omega_q - \frac{c}{2} \right) \left(\omega_r - \frac{c}{2} \right) - qr \right] - \left[m \left(\omega_p - \frac{c}{2} \right) - p \left(\omega_m - \frac{c}{2} \right) \right] \left[q \left(\omega_r - \frac{c}{2} \right) - r \left(\omega_q - \frac{c}{2} \right) \right] \right\} \quad (53) \end{aligned}$$

Also in this case we have chosen to write operators in normal ordered form. Since $H_{\text{int}}^{(2)}$ was derived as a classical object, it does not follow what the correct ordering of the operators is. A non-zero normal ordering constant would give a contribution to the Hamiltonian of the form

$$H_{\text{norm.ord.}} = \sum_n C_n \left((a_n^1)^\dagger a_n^1 + (a_n^2)^\dagger a_n^2 \right) \quad (54)$$

We assume in this paper that $C_n = 0$. Presumably one can argue for this on the same lines as in [20, 21]. Moreover, one can consider the single-magnon state $(a_n^1)^\dagger |0\rangle$ which, based on the general dispersion relation (2), should not receive $1/J$ corrections. This is consistent with $C_n = 0$. Finally, we shall see in Section 5 that we get agreement for the $|s\rangle$ and $|t\rangle$ string states with the Bethe ansatz assuming $C_n = 0$.

We now compute the energies of the states $|s\rangle$ and $|t\rangle$ (49)-(50). Consider first the state $|t\rangle = (a_n^1)^\dagger (a_{-n}^2)^\dagger |0\rangle$. To derive the mean value of (53) we need the following quantity

$$\langle 0 | (a_n^1) (a_{-n}^2) (a_{-m}^1)^\dagger a_p^1 (a_{-q}^2)^\dagger a_r^2 (a_n^1)^\dagger (a_{-n}^2)^\dagger | 0 \rangle = \delta_{m,-n} \delta_{p,n} \delta_{q,n} \delta_{r,-m}$$

so that the mean value of (53) contributing to (51) reads

$$\langle t | H_{\text{int}}^{(2)} | t \rangle = - \frac{\left[n^2 + \left(\omega_n - \frac{c}{2} \right)^2 \right]^2 + 4n^2 \left(\omega_n - \frac{c}{2} \right)^2}{R^2 c^3 \omega_n^2} \simeq - \frac{4n^4 \pi^4 \lambda'^2}{J} + \frac{16n^6 \pi^6 \lambda'^3}{J} + \mathcal{O}(\lambda'^4) \quad (55)$$

where λ' is defined in (25) and we used that $R^2 = 4\pi\sqrt{2\lambda}$.

To compute the second term in (51) we need to consider intermediate states that give a non vanishing matrix element for the $H_{\text{int}}^{(1)}$ given in (52). The only possible intermediate states that have this property are three oscillator states of the form $(a_{-p-q}^4)^\dagger (a_p^1)^\dagger (a_q^2)^\dagger |0\rangle$. Computing the matrix element is simple and we get

$$\sum_{|i\rangle} \frac{\left| \langle i | H_{\text{int}}^{(1)} | t \rangle \right|^2}{E_{|t\rangle}^{(0)} - E_{|i\rangle}^{(0)}} = \frac{1}{R^2 c} \sum_p \frac{\left[(\omega_{p+n} - \frac{c}{2}) \left(\omega_n - \frac{c}{2} \right) - (p+n)n \right]^2}{\omega_{p+n} \omega_n \Omega_p (\omega_{p+n} - \omega_n - \Omega_p)} + \frac{\left[\left(\omega_n - \frac{c}{2} \right)^2 - n^2 \right]^2}{R^2 c^3 \omega_n^2} \quad (56)$$

Using ζ -function regularization the first term vanishes, so that for the $\mathcal{O}(\frac{1}{R^2})$ correction to the energy of the state $|t\rangle$, we get, adding (55) and (56)

$$E_t^{(2)} = - \frac{\left[n^2 + \left(\omega_n - \frac{c}{2} \right)^2 \right]^2 + 4n^2 \left(\omega_n - \frac{c}{2} \right)^2}{R^2 c^3 \omega_n^2} + \frac{\left[\left(\omega_n - \frac{c}{2} \right)^2 - n^2 \right]^2}{R^2 c^3 \omega_n^2} \simeq - \frac{64n^6 \pi^6 \lambda'^3}{J} + \mathcal{O}(\lambda'^4) \quad (57)$$

It is interesting to note that for the state $|t\rangle$ the first finite-size correction appears at the order λ'^3 . In particular, that the finite-size correction is zero at order λ'^2 is due to a rather non-trivial cancelation of the mean-value contribution of (41) and the contribution coming from the $1/\sqrt{J}$ interaction term (40), which enters through a second-order perturbative energy correction and is regularized using ζ -function regularization.

Consider now the state $|s\rangle = (a_n^1)^\dagger (a_{-n}^1)^\dagger |0\rangle$. Since we have that

$$\langle 0 | a_n^1 a_{-n}^1 (a_{-m}^1)^\dagger a_p^1 (a_{-q}^2)^\dagger a_r^2 (a_n^1)^\dagger (a_{-n}^1)^\dagger | 0 \rangle = (\delta_{m,n} \delta_{p,-n} + \delta_{m,-n} \delta_{p,n}) (\delta_{q,n} \delta_{r,-n} + \delta_{r,n} \delta_{q,-n})$$

one gets

$$\langle s | H_{\text{int}}^{(2)} | s \rangle = - \frac{2 \left[(\omega_n - c) (4n^2 - c^2) - c^2 \omega_n \right]}{R^2 c^3 \omega_n} \simeq \frac{8n^2 \pi^2 \lambda'}{J} - \frac{56n^4 \pi^4 \lambda'^2}{J} + \frac{352n^6 \pi^6 \lambda'^3}{J} + \mathcal{O}(\lambda'^4) \quad (58)$$

To compute the second term in (51) we need to consider intermediate states of the form $(a_{-p-q}^4)^\dagger (a_p^1)^\dagger (a_q^1)^\dagger |0\rangle$. Computing the matrix element of (52), the second term in (51) gives the contribution

$$\sum_{|i\rangle} \frac{\left| \langle i | H_{\text{int}}^{(1)} | s \rangle \right|^2}{E_{|s\rangle}^{(0)} - E_{|i\rangle}^{(0)}} = \frac{1}{R^2 c} \sum_p \frac{\left[(\omega_{p+n} - \frac{c}{2}) \left(\omega_n - \frac{c}{2} \right) - (p+n)n \right]^2}{\omega_{p+n} \omega_n \Omega_p (\omega_{p+n} - \omega_n - \Omega_p)} - \frac{\left[\left(\omega_n - \frac{c}{2} \right)^2 + n^2 \right]^2}{R^2 c \omega_n^2 \Omega_{2n}^2} - \frac{\left[\left(\omega_n - \frac{c}{2} \right)^2 - n^2 \right]^2}{R^2 c^3 \omega_n^2} \quad (59)$$

where we have divided by 2 to avoid overcounting of intermediate states. Using ζ -function regularization the first term vanishes, so that for the $\mathcal{O}(\frac{1}{R^2})$ correction to the energy of the state $|s\rangle$, adding

(58) and (59), we get

$$\begin{aligned}
E_s^{(2)} &= -2 \frac{[(\omega_n - c)(4n^2 - c^2) - c^2 \omega_n]}{R^2 c^3 \omega_n} - \frac{[(\omega_n - \frac{c}{2})^2 + n^2]^2}{R^2 c \omega_n^2 \Omega_{2n}^2} - \frac{[(\omega_n - \frac{c}{2})^2 - n^2]^2}{R^2 c^3 \omega_n^2} \\
&\simeq \frac{8n^2 \pi^2 \lambda'}{J} - \frac{64n^4 \pi^4 \lambda'^2}{J} + \frac{448n^6 \pi^6 \lambda'^3}{J} + \mathcal{O}(\lambda'^4)
\end{aligned} \tag{60}$$

4 Low energy sigma-model expansion

In this section we consider the low energy sigma-model expansion in which $\Delta - J$ is small. In this way we zoom in to the $SU(2) \times SU(2)$ sector on the string side with $\lambda' = \lambda/J^2$ being small. To leading order we reproduce the result of [6] that the sigma-model consists of two Landau-Lifshitz models added together without interaction. We then move on to obtain the first and second order corrections in λ' to the leading sigma-model. We compare the energies of the $|s\rangle$ and $|t\rangle$ string states found in Section 3 for the λ' , λ'^2 and λ'^3 orders and find agreement.

The methods that we employ in this section have been developed in [23, 24, 28, 29, 30].

4.1 Expansion of sigma-model action

We want to extract the effective sigma-model description of the $SU(2) \times SU(2)$ sector, including corrections in λ' . Define

$$x^+ = \lambda' t, \quad x^- = \delta - \frac{1}{2}t \tag{61}$$

with

$$\lambda' \equiv \frac{\lambda}{J^2} \tag{62}$$

Then the charges are

$$\frac{E}{\lambda'} = \frac{\Delta - J}{\lambda'} = -P_+ = i\partial_{x^+}, \quad P_- = -i\partial_{x^+} = 2J \tag{63}$$

We see that taking the $\lambda' \rightarrow 0$ limit means that $\Delta - J \rightarrow 0$ which means that we keep the modes of the $SU(2) \times SU(2)$ sector dynamical, while the other modes become non-dynamical in this limit. Naively, this leads to the reasoning that one can set $\rho = 0$ and $\psi = 0$ in the $\text{AdS}_4 \times \mathbb{CP}^3$ background (5) with the \mathbb{CP}^3 metric given by (11). However, as we shall see in the following the field ψ does couple to the modes of the $SU(2) \times SU(2)$ sector even though it becomes non-dynamical in the $\lambda' \rightarrow 0$ limit.

Consider therefore the $\text{AdS}_4 \times \mathbb{CP}^3$ metric given by (5) and (11) with $\rho = 0$ and in terms of the x^+, x^- coordinates (61)

$$\begin{aligned}
ds^2 &= R^2 \left[-\frac{1}{4\lambda'^2} \sin^2 \psi (dx^+)^2 + \frac{1}{4} d\psi^2 + \cos^2 \psi (\lambda'^{-1} dx^+ + dx^- + \omega)(dx^- + \omega) \right. \\
&\quad \left. + \frac{1 - \sin \psi}{8} d\Omega_2^2 + \frac{1 + \sin \psi}{8} d\Omega_2'^2 \right]
\end{aligned} \tag{64}$$

The idea in the following is that ψ as expected is non-dynamical in the $\lambda' \rightarrow 0$ limit, however, one has to include it. We show that in the $\lambda' \rightarrow 0$ limit ψ acts as a Lagrange multiplier, and solving the constraint associated to ψ gives extra terms to the effective sigma-model.

We consider the bosonic sigma-model Lagrangian

$$\mathcal{L} = -\frac{1}{2} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \tag{65}$$

with the Virasoro constraints

$$G_{\mu\nu} (\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) = 0 \tag{66}$$

with $G_{\mu\nu}$ being the metric (64). Define for convenience

$$A \equiv -h^{00}, \quad B \equiv h^{01} \quad (67)$$

Since the determinant of $h^{\alpha\beta}$ is -1 we have $h^{11} = (1 - B^2)/A$. For $\lambda' \rightarrow 0$ we have that $A = 1$ and $B = 0$. Define

$$S_{\alpha\beta} \equiv G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \quad (68)$$

We can now write the Lagrangian as

$$\mathcal{L} = \frac{A}{2} S_{00} - B S_{01} - \frac{1 - B^2}{2A} S_{11} \quad (69)$$

and the Virasoro constraints as

$$\begin{aligned} (1 + B^2) S_{00} + \frac{2B(1 - B^2)}{A} S_{01} + \frac{(1 - B^2)^2}{A^2} S_{11} &= 0 \\ ABS_{00} + 2(1 - B^2) S_{01} - \frac{B(1 - B^2)}{A} S_{11} &= 0 \end{aligned} \quad (70)$$

Our gauge choice is

$$x^+ = \kappa\tau \quad (71)$$

$$2\pi p_- = \frac{\partial \mathcal{L}}{\partial \partial_\tau x^-} = \text{const.}, \quad \frac{\partial \mathcal{L}}{\partial \partial_\sigma x^-} = 0 \quad (72)$$

Thus, we are not fixing the world-sheet metric in this gauge choice, but rather that the angular momentum J is evenly distributed along the string [24]. We have

$$2\pi p_- = R^2 \cos^2 \psi \left[\frac{A\kappa}{2\lambda'} + A(\partial_\tau x^- + \omega_\tau) - B(x^{-'} + \omega_\sigma) \right] \quad (73)$$

The ψ field will be seen to be a non-dynamical field, thus it should be considered here as a Lagrange-multiplier. For $\lambda' \rightarrow 0$ we require that $\psi \rightarrow 0$. The dominating term for $\lambda' \rightarrow 0$ is therefore

$$2\pi p_- = \frac{R^2 \kappa}{2\lambda'} \quad (74)$$

From this we obtain

$$2J = P_- = \int_0^{2\pi} d\sigma p_- = \frac{R^2 \kappa}{2\lambda'} = \frac{2\pi\sqrt{2\lambda}\kappa}{\lambda'} \quad (75)$$

where we used that $R^2 = 4\pi\sqrt{2\lambda}$. We see from this that

$$\kappa = \frac{\sqrt{\lambda'}}{\pi\sqrt{2}} \quad (76)$$

Thus κ goes like $\sqrt{\lambda'}$. This means that $\kappa \rightarrow 0$ for $\lambda' \rightarrow 0$. Write now

$$\partial_\tau x^\mu = \kappa \dot{x}^\mu \quad (77)$$

Then we should keep fixed \dot{x}^μ in the $\kappa \rightarrow 0$ limit, since that corresponds to the correct energy scale. Define therefore the rescaled world-sheet time $\tilde{\tau}$ as $\tilde{\tau} = \kappa\tau$ so that we have $\dot{x}^\mu = \partial_{\tilde{\tau}} x^\mu$.

Using the metric (64) we compute

$$\begin{aligned} S_{00} = R^2 \kappa^2 \left[-\frac{1}{4\lambda'^2} \sin^2 \psi + \frac{1}{4} \dot{\psi}^2 + \cos^2 \psi \left(\frac{1}{\lambda'} + \dot{x}^- + \dot{\omega} \right) (\dot{x}^- + \dot{\omega}) \right. \\ \left. + \frac{1 - \sin \psi}{8} (\dot{\theta}_1^2 + \cos^2 \theta_1 \dot{\varphi}_1^2) + \frac{1 - \sin \psi}{8} (\dot{\theta}_2^2 + \cos^2 \theta_2 \dot{\varphi}_2^2) \right] \end{aligned} \quad (78)$$

$$S_{01} = R^2 \kappa \left[\frac{1}{4} \dot{\psi} \psi' + \cos^2 \psi \left(\frac{1}{2\lambda'} + \dot{x}^- + \dot{\omega} \right) (x^{-'} + \omega_\sigma) \right. \\ \left. + \frac{1 - \sin \psi}{8} (\dot{\theta}_1 \theta'_1 + \cos^2 \theta_1 \dot{\varphi}_1 \varphi'_1) + \frac{1 + \sin \psi}{8} (\dot{\theta}_2 \theta'_2 + \cos^2 \theta_2 \dot{\varphi}_2 \varphi'_2) \right] \quad (79)$$

$$S_{11} = R^2 \left[\frac{1}{4} \psi'^2 + \cos^2 \psi (x^{-'} + \omega_\sigma)^2 + \frac{1 - \sin \psi}{8} (\theta_1'^2 + \cos^2 \theta_1 \varphi_1'^2) + \frac{1 + \sin \psi}{8} (\theta_2'^2 + \cos^2 \theta_2 \varphi_2'^2) \right] \quad (80)$$

To find the effective action we should solve the two Virasoro constraints (70) and the two gauge conditions (72) (with S_{00} , S_{01} and S_{11} as in Eqs. (78)-(80)) to obtain \dot{x}^- , $x^{-'}$, A and B in terms of the transverse fields and their derivatives. This we do order by order in κ . A convenient way to do this is to first solve the two gauge conditions (72) to find \dot{x}^- and $x^{-'}$ in terms of A , B and the transverse fields. This gives

$$\dot{x}^- = -\dot{\omega} - \frac{1}{2\lambda'} + \frac{1 - B^2}{2A\lambda' \cos^2 \psi}, \quad x^{-'} = -\omega_\sigma - \frac{B\kappa}{2\lambda' \cos^2 \psi} \quad (81)$$

We subsequently plug this into the Virasoro constraints (70) to solve for A and B in terms of the transverse fields and their derivatives. To this end we expand A and B as follows

$$A = 1 + \kappa^2 A_1 + \kappa^4 A_2 + \dots, \quad B = \kappa^3 B_1 + \kappa^5 B_2 + \dots \quad (82)$$

We furthermore make the following expansion of ψ

$$\psi = \kappa^2 \psi_1 + \kappa^4 \psi_2 + \dots \quad (83)$$

We now solve the Virasoro constraints (70) order by order in κ . We get

$$A_1 = \pi^4 \sum_{i=1}^2 (\vec{n}'_i)^2, \quad B_1 = 2\pi^4 \sum_{i=1}^2 \dot{\vec{n}}_i \cdot \vec{n}'_i \quad (84)$$

$$A_2 = \frac{\psi_1^2}{2} - \pi^4 \psi_1 [(\vec{n}'_1)^2 - (\vec{n}'_2)^2] + \pi^4 [(\dot{\vec{n}}_1)^2 + (\dot{\vec{n}}_2)^2] - \frac{\pi^8}{2} [(\vec{n}'_1)^2 + (\vec{n}'_2)^2]^2 \quad (85)$$

$$B_2 = -2\pi^4 \psi_1 [\dot{\vec{n}}_1 \cdot \vec{n}'_1 - \dot{\vec{n}}_2 \cdot \vec{n}'_2] - 2\pi^8 [(\vec{n}'_1)^2 + (\vec{n}'_2)^2] [\dot{\vec{n}}_1 \cdot \vec{n}'_1 + \dot{\vec{n}}_2 \cdot \vec{n}'_2] \quad (86)$$

where we here and in the following simplify our expressions by using the two unit vector fields $\vec{n}_i(\tilde{\tau}, \sigma)$, $i = 1, 2$, parameterized as

$$\vec{n}_i = (\cos \theta_i \cos \varphi_i, \cos \theta_i \sin \varphi_i, \sin \theta_i) \quad (87)$$

We now plug in \dot{x}^- , $x^{-'}$, A and B from (81) and (84)-(86) into the gauge fixed Lagrangian

$$\mathcal{L}_g = \mathcal{L} - 2\pi\kappa p_- \dot{x}^- \quad (88)$$

This gives

$$\mathcal{L}_g = \mathcal{L}_0 + \lambda' \mathcal{L}_1 + \lambda'^2 \mathcal{L}_2 + \dots \quad (89)$$

with

$$\mathcal{L}_0 = \frac{R^2}{16\pi^2} \sum_{i=1}^2 \left[\sin \theta_i \dot{\varphi}_i - \pi^2 (\vec{n}'_i)^2 \right] \quad (90)$$

$$\mathcal{L}_1 = \frac{R^2}{64\pi^2} \left[\sum_{i=1}^2 \left(2(\dot{\vec{n}}_i)^2 + \pi^4 (\vec{n}'_i)^4 \right) + 2\pi^4 (\vec{n}'_1)^2 (\vec{n}'_2)^2 + 2\psi_1 [(\vec{n}'_1)^2 - (\vec{n}'_2)^2] - \frac{\psi_1^2}{\pi^4} \right] \quad (91)$$

$$\begin{aligned}
\mathcal{L}_2 = & \frac{R^2}{64} \left\{ \frac{\psi_2}{\pi^4} \left[(\vec{n}_1)^2 - (\vec{n}_2)^2 - \frac{\psi_1}{\pi^4} \right] - \frac{2\psi_1'}{\pi^4} - \frac{\psi_1}{2\pi^4} \left[(\dot{\vec{n}}_1)^2 - (\dot{\vec{n}}_2)^2 + \pi^4 [(\vec{n}_1')^4 - (\vec{n}_2')^4] \right] \right. \\
& + \frac{\psi_1^2}{2\pi^4} [(\vec{n}_1')^2 + (\vec{n}_2')^2] - \frac{\pi^4}{2} [(\vec{n}_1')^2 + (\vec{n}_2')^2]^3 - 2(\dot{\vec{n}}_1 \cdot \vec{n}_1' + \dot{\vec{n}}_2 \cdot \vec{n}_2')^2 \\
& \left. + [(\dot{\vec{n}}_1)^2 + (\dot{\vec{n}}_2)^2][(\vec{n}_1')^2 + (\vec{n}_2')^2] \right\} \quad (92)
\end{aligned}$$

We see that \mathcal{L}_0 is the sum of two Landau-Lifshitz models, reproducing the result already found in [6]. In \mathcal{L}_1 we see that the first part is non-interacting in the two $SU(2)$'s, then there is a interaction term and then a coupling to ψ . We see that ψ_1 appears as a Lagrange-multiplier, *i.e.* it is not a dynamical field. The EOM for ψ_1 is found to be satisfied provided

$$\psi_1 = \pi^4 [(\vec{n}_1')^2 - (\vec{n}_2')^2] \quad (93)$$

Inserting this into \mathcal{L}_1 , we get

$$\mathcal{L}_1 = \frac{R^2}{32\pi^2} \sum_{i=1}^2 \left[(\dot{\vec{n}}_i)^2 + \pi^4 (\vec{n}_i')^4 \right] \quad (94)$$

We see that there are no interaction terms and the two $SU(2)$'s appear symmetrically.

For \mathcal{L}_2 we should first substitute in ψ_1 from (93). This gives

$$\begin{aligned}
\mathcal{L}_2 = & \frac{R^2}{64} \left[-2(\dot{\vec{n}}_1 \cdot \vec{n}_1' + \dot{\vec{n}}_2 \cdot \vec{n}_2')^2 + 2(\dot{\vec{n}}_1)^2 (\vec{n}_2')^2 + 2(\dot{\vec{n}}_2)^2 (\vec{n}_1')^2 \right. \\
& \left. - \pi^4 \left((\vec{n}_1')^6 + (\vec{n}_2')^6 + (\vec{n}_1')^2 (\vec{n}_2')^4 + (\vec{n}_1')^4 (\vec{n}_2')^2 + 8(\vec{n}_1' \cdot \vec{n}_1'' - \vec{n}_2' \cdot \vec{n}_2'')^2 \right) \right] \quad (95)
\end{aligned}$$

We see now that ψ_2 has disappeared from the Lagrangian after substituting ψ_1 . We also notice that there are interaction terms in (95).

We now want to eliminate the time derivatives in \mathcal{L}_1 and \mathcal{L}_2 . To do this we should perform a field redefinition, following [24]

$$\vec{n}_i \rightarrow \vec{n}_i + \lambda' \vec{p}_i + \lambda'^2 \vec{q}_i \quad (96)$$

in terms of \vec{n}_i and their derivatives. By choosing \vec{p}_i and \vec{q}_i it is possible to eliminate the time-derivatives. Write first \mathcal{L}_1 and \mathcal{L}_2 as

$$\mathcal{L}_1 = \sum_{i=1}^2 \vec{u}_i \cdot \frac{\delta \mathcal{L}_0}{\delta \vec{n}_i} + (\mathcal{L}_1)_0, \quad \mathcal{L}_2 = \sum_{i=1}^2 \vec{v}_i \cdot \frac{\delta \mathcal{L}_0}{\delta \vec{n}_i} + (\mathcal{L}_2)_0 \quad (97)$$

where $(\mathcal{L}_i)_0$ are \mathcal{L}_i without time-derivatives obtained by using the leading EOM

$$\frac{\delta \mathcal{L}_0}{\delta \vec{n}_i} = 0 \quad (98)$$

One can check that we can use the same field redefinition as in [24]. This redefinition consists in choosing $\vec{p}_i = -\vec{u}_i$ and \vec{q}_i is furthermore chosen such that we get the new Lagrangian

$$\mathcal{L}_g = \mathcal{L}_0 + \lambda' (\mathcal{L}_1)_0 + \lambda'^2 \hat{\mathcal{L}}_2 \quad (99)$$

with

$$\hat{\mathcal{L}}_2 = (\mathcal{L}_2)_0 - \sum_{i=1}^2 \frac{\delta (\mathcal{L}_1)_0}{\delta \vec{n}_i} \cdot (\vec{u}_i)_0 + \sum_{i=1}^2 \sum_{a,b=1}^3 \left(\frac{\delta^2 \mathcal{L}_0}{\delta (n_i)_a \delta (n_i)_b} \right)_0 (u_{i,a})_0 (u_{i,b})_0 \quad (100)$$

Notice that the last two terms only involve \mathcal{L}_0 and \mathcal{L}_1 . Since \vec{n}_1 and \vec{n}_2 are decoupled in \mathcal{L}_0 and \mathcal{L}_1 the last two terms do not contain any interaction terms between the two two-spheres. This is also the reason why we can directly use the field redefinition of [24].

The leading EOM is obtained from

$$\frac{\delta \mathcal{L}_0}{\delta \vec{n}_i} = \frac{R^2}{16\pi^2} \left(\vec{n}_i \times \dot{\vec{n}}_i + 2\pi^2 (\vec{n}_i'')_{\perp} \right) \quad (101)$$

with

$$(\vec{n}_i'')_{\perp} = \vec{n}_i'' + \vec{n}(\vec{n}')^2, \quad ((\vec{n}_i'')_{\perp})^2 = (\vec{n}_i'')^2 - (\vec{n}_i')^4 \quad (102)$$

Thus the leading EOM is

$$\vec{n}_i \times \dot{\vec{n}}_i = -2\pi^2 (\vec{n}_i'')_{\perp} \quad (103)$$

giving

$$(\dot{\vec{n}}_i)^2 = 4\pi^4 ((\vec{n}_i'')_{\perp})^2 = 4\pi^4 [(\vec{n}_i'')^2 - (\vec{n}_i')^4], \quad \vec{n}_i' \cdot \dot{\vec{n}}_i = -2\pi^2 \vec{n}_i \cdot (\vec{n}_i' \times \vec{n}_i'') \quad (104)$$

This gives the following on-shell evaluations of \mathcal{L}_1 and \mathcal{L}_2

$$(\mathcal{L}_1)_0 = \frac{\pi^2 R^2}{8} \sum_{i=1}^2 \left[(\vec{n}_i'')^2 - \frac{3}{4} (\vec{n}_i')^4 \right] \quad (105)$$

$$\begin{aligned} (\mathcal{L}_2)_0 = & \frac{\pi^4 R^2}{64} \left\{ \sum_{i=1}^2 \left(7(\vec{n}_i')^6 - 8(\vec{n}_i')^2 (\vec{n}_i'')^2 \right) + 8[(\vec{n}_1')^2 (\vec{n}_2'')^2 + (\vec{n}_2')^2 (\vec{n}_1'')^2] \right. \\ & + 16(\vec{n}_1' \cdot \vec{n}_1'')(\vec{n}_2' \cdot \vec{n}_2'') - 9[(\vec{n}_1')^2 (\vec{n}_2')^4 + (\vec{n}_2')^2 (\vec{n}_1')^4] \\ & \left. - 16(\vec{n}_1 \cdot (\vec{n}_1' \times \vec{n}_1''))(\vec{n}_2 \cdot (\vec{n}_2' \times \vec{n}_2'')) \right\} \quad (106) \end{aligned}$$

We now compute

$$\widehat{\mathcal{L}}_2 - (\mathcal{L}_2)_0 = \frac{\pi^4 R^2}{2} \sum_{i=1}^2 \left[-(\vec{n}_i''')^2 + 2(\vec{n}_i')^2 (\vec{n}_i'')^2 + 12(\vec{n}_i' \cdot \vec{n}_i'')^2 - (\vec{n}_i')^6 \right] \quad (107)$$

Using this, we obtain

$$\begin{aligned} \widehat{\mathcal{L}}_2 = & \frac{\pi^4 R^2}{2} \left\{ \sum_{i=1}^2 \left(-(\vec{n}_i''')^2 - \frac{7}{4} (\vec{n}_i')^2 (\vec{n}_i'')^2 + 12(\vec{n}_i' \cdot \vec{n}_i'')^2 - \frac{25}{32} (\vec{n}_i')^6 \right) \right. \\ & + \frac{1}{4} [(\vec{n}_1')^2 (\vec{n}_2'')^2 + (\vec{n}_2')^2 (\vec{n}_1'')^2] - \frac{9}{32} [(\vec{n}_1')^2 (\vec{n}_2')^4 + (\vec{n}_2')^2 (\vec{n}_1')^4] \\ & \left. + \frac{1}{2} (\vec{n}_1' \cdot \vec{n}_1'')(\vec{n}_2' \cdot \vec{n}_2'') - \frac{1}{2} (\vec{n}_1 \cdot (\vec{n}_1' \times \vec{n}_1''))(\vec{n}_2 \cdot (\vec{n}_2' \times \vec{n}_2'')) \right\} \quad (108) \end{aligned}$$

The final sigma-model action is

$$I = \frac{4\pi J}{R^2} \int d\tilde{\tau} d\sigma \left[\mathcal{L}_0 + \lambda' (\mathcal{L}_1)_0 + \lambda'^2 \widehat{\mathcal{L}}_2 \right] \quad (109)$$

Thus, the action with time-derivatives only in the leading part is given by (109) along with (90), (105) and (108). We notice again that for the leading part \mathcal{L}_0 , corresponding to order λ' , and the first correction $(\mathcal{L}_1)_0$, corresponding to order λ'^2 , there are no interactions between the two two-spheres. In fact \mathcal{L}_0 and $(\mathcal{L}_1)_0$ are equivalent to Lagrangians found for the $SU(2)$ sector of $\text{AdS}_5 \times S^5$ in [23, 24]. Instead for the second order correction $\widehat{\mathcal{L}}_2$, corresponding to λ'^3 , there are interactions between the two two-spheres, and also the part acting only on a single $SU(2)$ is different from that found in [23, 24] for the $SU(2)$ sector of $\text{AdS}_5 \times S^5$.

4.2 Computation of finite-size correction to energies

In the following we compute the finite size correction to the energies of the two string states $|s\rangle$ and $|t\rangle$ considered in Section 3 using the action (109). In order to accomplish this, we first need to write down the Hamiltonian. We begin by observing that the conjugate momenta to φ_i are

$$p_{\varphi_i} = J \sin \theta_i \quad (110)$$

Notice that we have left out a factor 4π in front of the action. With this, we can write the action (109) as

$$I = \frac{J}{4\pi} \int d\tilde{\tau} d\sigma \left[\frac{1}{J} \sum_{i=1}^2 p_{\varphi_i} \dot{\varphi}_i - \mathcal{H}_0 - \lambda' \mathcal{H}_1 - \lambda'^2 \mathcal{H}_2 \right] \quad (111)$$

with

$$\mathcal{H}_0 = \pi^2 \sum_{i=1}^2 (\vec{n}'_i)^2, \quad \mathcal{H}_1 = -2\pi^4 \sum_{i=1}^2 \left[(\vec{n}''_i)^2 - \frac{3}{4} (\vec{n}'_i)^4 \right] \quad (112)$$

$$\begin{aligned} \mathcal{H}_2 = & 8\pi^6 \left\{ \sum_{i=1}^2 \left((\vec{n}'''_i)^2 + \frac{7}{4} (\vec{n}'_i)^2 (\vec{n}''_i)^2 - 12 (\vec{n}'_i \cdot \vec{n}''_i)^2 + \frac{25}{32} (\vec{n}'_i)^6 \right) - \frac{1}{4} [(\vec{n}'_1)^2 (\vec{n}'_2)^2 + (\vec{n}'_2)^2 (\vec{n}'_1)^2] \right. \\ & \left. + \frac{9}{32} [(\vec{n}'_1)^2 (\vec{n}'_2)^4 + (\vec{n}'_2)^2 (\vec{n}'_1)^4] - \frac{1}{2} (\vec{n}'_1 \cdot \vec{n}'_1) (\vec{n}'_2 \cdot \vec{n}'_2) + \frac{1}{2} (\vec{n}'_1 \cdot (\vec{n}'_1 \times \vec{n}'_1)) (\vec{n}'_2 \cdot (\vec{n}'_2 \times \vec{n}'_2)) \right\} \quad (113) \end{aligned}$$

The Hamiltonian is thus

$$H = \frac{J}{4\pi} \int d\sigma \left[\mathcal{H}_0 + \lambda' \mathcal{H}_1 + \lambda'^2 \mathcal{H}_2 \right] \quad (114)$$

where \vec{n}_i in terms of φ_i and p_{φ_i} is

$$\vec{n}_i = \left(\sqrt{1 - \frac{p_{\varphi_i}^2}{J^2}} \cos \varphi_i, \sqrt{1 - \frac{p_{\varphi_i}^2}{J^2}} \sin \varphi_i, \frac{p_{\varphi_i}}{J} \right) \quad (115)$$

To compute the finite-size correction to a string state we want to zoom in to $(\theta_i, \varphi_i) = (0, 0)$. We do this by defining

$$x_i = \sqrt{J} \varphi_i, \quad y_i = \sqrt{J} \theta_i \quad (116)$$

The conjugate momenta for x_i are

$$p_i = \sqrt{J} \sin \theta_i \quad (117)$$

We now write the Hamiltonian up to $1/J^2$ corrections in terms of the new variables. We get that

$$H = H_0 + \lambda' H_1 + \lambda'^2 H_2 \quad (118)$$

with

$$H_0 = \frac{\pi}{4} \sum_{i=1}^2 \int_0^{2\pi} \left\{ x_i'^2 + p_i'^2 + \frac{1}{J} \left[p_i'^2 (p_i'^2 - x_i'^2) \right] \right\} \quad (119)$$

$$H_1 = -\frac{\pi^3}{2} \sum_{i=1}^2 \int_0^{2\pi} \left\{ x_i''^2 + p_i''^2 + \frac{1}{J} \left[(p_i'')^2 (p_i''^2 - x_i''^2) + 2p_i p_i'' (p_i'^2 + x_i'^2) - 4p_i p_i' x_i' x_i'' \right] \right\} \quad (120)$$

$$\begin{aligned} H_2 = & \pi^5 \sum_{i=1}^2 \int_0^{2\pi} \left\{ 2(x_i''')^2 + 2(p_i''')^2 - \frac{1}{2J} \left[4(p_i'^2 ((x_i''')^2 + p_i''')^2) + 8(x_i')^3 x_i''' \right. \right. \\ & + 24p_i p_i' (x_i'' x_i''' - p_i'' p_i''') + 24p_i x_i' (p_i'' x_i''' - x_i'' p_i''') + 24x_i' p_i' (p_i' x_i''' + x_i'' p_i'') \\ & \left. \left. - 7((p_i')^2 (x_i'')^2 + (x_i')^2 (p_i'')^2) + 5((x_i')^2 (x_i'')^2 + (p_i')^2 (p_i'')^2) \right] \right\} + \bar{H}_2 \quad (121) \end{aligned}$$

where \bar{H}_2 is given by

$$\begin{aligned} \bar{H}_2 = & -\frac{\pi^5}{J} \int_0^{2\pi} \left\{ \frac{1}{2} [(x_1'')^2 + (p_1'')^2] [(x_2')^2 + (p_2')^2] + [(x_2'')^2 + (p_2'')^2] [(x_1')^2 + (p_1')^2] \right. \\ & \left. - (x_1''x_2'' - p_1''p_2'') (p_1'p_2' - x_1'x_2') + (x_1''p_2'' + p_1''x_2'') (p_1'x_2' + x_1'p_2') \right\} \end{aligned} \quad (122)$$

It is interesting to notice that the only part of the above Hamiltonian with interactions between the two two-spheres is in \bar{H}_2 . This means that the leading interaction between the two two-spheres appear at order λ'^3/J in agreement with what we have seen in Section 3.

From the EOM we obtain the following mode expansions

$$x_i(t, \sigma) = \sum_{n=-\infty}^{\infty} (a_n^i e^{-i\bar{\omega}_n t + i n \sigma} + a_n^{i\dagger} e^{i\bar{\omega}_n t - i n \sigma}) \quad (123)$$

$$p_i(t, \sigma) = -i \sum_{n=-\infty}^{\infty} (a_n^i e^{-i\bar{\omega}_n t + i n \sigma} - a_n^{i\dagger} e^{i\bar{\omega}_n t - i n \sigma}) \quad (124)$$

where

$$\bar{\omega}_n = 2\pi^2(n^2 - 2\pi^2\lambda'n^4 + 8\pi^4\lambda'^2 n^6) \quad (125)$$

which coincides with the expansion up to $\mathcal{O}(\lambda'^4)$ of $\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - \frac{1}{2}$. By imposing $[a_m^i, (a_n^j)^\dagger] = \delta_{mn} \delta_{ij}$ we obtain the standard canonical commutation relation $[x_i(t, \sigma), p_j(t, \sigma')] = i\delta_{ij} \delta(\sigma - \sigma')$.

From Eqs. (119)-(121) we see that we obtain the free spectrum

$$E_0 = 2\pi^2\lambda' \sum_{n \in \mathbb{Z}} (n^2 - 2\pi^2\lambda'n^4 + 8\pi^4\lambda'^2 n^6) (M_n^1 + M_n^2), \quad \sum_{n \in \mathbb{Z}} n (M_n^1 + M_n^2) = 0 \quad (126)$$

which coincides with the expansion up to $\mathcal{O}(\lambda'^4)$ of the spectrum (48), (47).

We now want to compute the $1/J$ corrections to the free spectrum. These are obtained from the terms in Eqs. (119)-(121) which are quartic in the fields. Considering the state $|s\rangle = a_n^{1\dagger} a_{-n}^{1\dagger} |0\rangle$, we obtain

$$E - E_0 = \frac{8n^2\pi^2\lambda'}{J} - \frac{64n^4\pi^4\lambda'^2}{J} + \frac{448n^6\pi^6\lambda'^3}{J} \quad (127)$$

which is in perfect agreement with the expansion of the energy (60) of the state $|s\rangle$ computed in Section 3. Moreover, considering the state $|t\rangle = a_n^{1\dagger} a_{-n}^{2\dagger} |0\rangle$, we obtain the energy

$$E - E_0 = -\frac{64n^6\pi^6\lambda'^3}{J} \quad (128)$$

This is also in perfect agreement with the expansion of the energy (57) of the state $|t\rangle$ computed in Section 3.

It is interesting to notice that the absence of interactions between the two two-spheres at order λ'^2 here is due to the non-trivial coupling with the non-dynamical field ψ , while in Section 3 it is also due to the field ψ but there ψ contributes through a second order perturbative correction which is regularized using ζ -function regularization.

5 Comparison with all-loop Bethe ansatz

In the recent paper [12] Gromov and Vieira proposed a set of all loop Bethe equations for the full asymptotic spectrum of the AdS_4/CFT_3 duality. We shall provide here the explicit expressions for the rapidities and the dressing factors for these Bethe equations in the $SU(2) \times SU(2)$ sector in the strong

coupling regime, $\lambda \gg 1$. We shall then solve perturbatively the Bethe equations constructed in this way and derive the first non-trivial finite size corrections. These can then be compared to the results we found from the explicit quantum calculations on two oscillator states both from the string theory sigma model and from the corresponding Landau-Lifshitz model.

For the $SU(2) \times SU(2)$ sector in the strong-coupling region $\lambda \gg 1$ the Bethe equations read [12]

$$e^{ip_k J} = \prod_{j=1, j \neq k}^{K_p} S(p_k, p_j) \prod_{j=1}^{K_p} \sigma(p_k, p_j) \prod_{j=1}^{K_q} \sigma(p_k, q_j) \quad (129)$$

$$e^{iq_k J} = \prod_{j=1, j \neq k}^{K_p} S(q_k, q_j) \prod_{j=1}^{K_p} \sigma(q_k, q_j) \prod_{j=1}^{K_q} \sigma(q_k, p_j) \quad (130)$$

$$S(p_k, p_j) = \frac{\Phi(p_k) - \Phi(p_j) + i}{\Phi(p_k) - \Phi(p_j) - i} \quad (131)$$

The explicit form of the rapidities $\Phi(p)$ and of the dressing factor $\sigma(q_k, q_j)$ for this sector can be constructed along the lines of those found in the AdS_5/CFT_4 duality [31, 18, 19]. The rapidities are

$$\Phi(p_j) = \cot \frac{p_j}{2} \sqrt{\frac{1}{4} + h(\lambda) \sin^2 \frac{p_j}{2}} \quad (132)$$

where, here, at strong coupling, $h(\lambda) = 2\lambda$ [5, 6]. The relevant part of the dressing factor in terms of the conserved charges $Q_r(p)$ reads

$$\sigma(p_j, p_l) = \exp \left\{ 2i \sum_{r=0}^{\infty} \left(\frac{h(\lambda)}{16} \right)^{r+2} [Q_{r+2}(p_j) Q_{r+3}(p_l) - Q_{r+2}(p_l) Q_{r+3}(p_j)] \right\} \quad (133)$$

where we can write

$$Q_r(p_j) = \frac{2 \sin(\frac{r-1}{2} p_j)}{r-1} \left(\frac{\sqrt{\frac{1}{4} + h(\lambda) \sin^2 \frac{p_j}{2}} - \frac{1}{2}}{\frac{h(\lambda)}{4} \sin \frac{p_l}{2}} \right)^{r-1} \quad (134)$$

We then have the dispersion relation

$$\begin{aligned} E = \Delta - J &= \frac{h(\lambda)}{8} \left(\sum_{j=1}^{K_p} Q_2(p_j) + \sum_{j=1}^{K_q} Q_2(p_j) \right) \\ &= \sum_{j=1}^{K_p} \left(\sqrt{\frac{1}{4} + h(\lambda) \sin^2 \frac{p_j}{2}} - \frac{1}{2} \right) + \sum_{j=1}^{K_q} \left(\sqrt{\frac{1}{4} + h(\lambda) \sin^2 \frac{q_j}{2}} - \frac{1}{2} \right) \end{aligned} \quad (135)$$

We will now discuss the two magnon case in the $AdS_4 \times \mathbb{CP}^3$ theory and solve the Bethe equations. These can be solved perturbatively in λ' and J following the procedure adopted for example in [32]. When one magnon is in one $SU(2)$ sector and the other magnon is in the other ⁵, the scattering matrix becomes trivial and the momentum is just given by the dressing phase. At the first non trivial order in λ' we get

$$e^{ip_1 J} = e^{i \frac{\lambda'^2}{32} [Q_2(p_1) Q_3(q_1) - Q_3(p_1) Q_2(q_1)]} \quad (136)$$

with $q_1 = -p_1$ from the momentum constraint. Since the scattering matrix is just 1 in this case, quite interestingly, the momentum starts to receive corrections only at the order λ'^2 and this will provide

⁵This situation corresponds to the state $|t\rangle$ on the string theory side, so we label the corresponding energy/scaling dimension as E_t .

a non vanishing contribution to the finite size correction to the energy only at the order λ'^3 . This is analogous to what we found on the string theory side both from computing curvature corrections to the Penrose limit in Section 3 and by considering a low energy expansion of the string theory sigma model in Section 4.

For the momentum we can consider an ansatz of the form

$$p_1 = \frac{2\pi n}{J} + \frac{a\lambda'^2}{J^2} + \mathcal{O}\left(\lambda'^3, \frac{1}{J^2}\right) \quad (137)$$

where a is a parameter that will be determined by requiring that the Bethe equations are satisfied at this order. Plugging (137) into (136) and expanding for small λ and large J it is easy to determine a as $a = -16\pi^5 n^5$. Using this result for the momentum in the dispersion relation (135) we get for the first non trivial finite size correction

$$E_t = 4n^2\pi^2\lambda' - 8n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 - \frac{64n^6\pi^6\lambda'^3}{J} + \mathcal{O}\left(\frac{\lambda'}{J^2}\right) \quad (138)$$

where we have written only the leading terms in $\lambda' = \frac{\lambda}{J^2}$ and the first finite size correction. This is provided by the last term. The result precisely coincides with the one found in Sections 3 and 4, see eq.s (57) and (128). We see here that the reason why the finite size correction only starts at three loops is basically due to the fact that the only non trivial factor in the Bethe equations is the dressing phase.

We solved the Bethe equations up to the order λ'^8 and we found perfect agreement with the string theory result for the energy E_t . We can then conclude that the dispersion relation up to the first order in $1/J$ for two magnons, one in each $SU(2)$ sector, is

$$E_t = 2\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 1 - \frac{\lambda'}{J} \frac{4\pi^2 n^2}{\frac{1}{4} + 2\pi^2 n^2 \lambda'} \left(\frac{1}{2} + 2\pi^2 n^2 \lambda' - \sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} \right) \quad (139)$$

in the limit of large λ with $\lambda' = \lambda/J^2$ fixed.

The solution of the Bethe equations for the two magnon case, when these belong to the same $SU(2)$ sector ⁶, can be obtained in the same way. The Bethe equations in this case read

$$e^{ip_1 J} = \frac{\Phi(p_1) - \Phi(p_2) + i}{\Phi(p_1) - \Phi(p_2) - i} e^{i\frac{\lambda'^2}{32} [Q_2(p_1)Q_3(p_2) - Q_3(p_1)Q_2(p_2)]} \quad (140)$$

where, from the momentum constraint, $p_2 = -p_1$. The correct ansatz for the expansion of the momentum now is

$$p_1 = \frac{2\pi n}{J-1} + \frac{a\lambda'}{J^2} + \frac{b\lambda'^2}{J^2} + \mathcal{O}\left(\lambda'^3, \frac{1}{J^2}\right) \quad (141)$$

which substituted into the Bethe equations (140) provides the following solutions for the parameters a and b : $a = -8\pi^3 n^3$, $b = 32n^5\pi^5$. Plugging the solution for the momentum back into the dispersion relation we get

$$E_s = 4n^2\pi^2\lambda' - 8n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 + \frac{8n^2\pi^2\lambda'}{J} - \frac{64n^4\pi^4\lambda'^2}{J} + \frac{448n^6\pi^6\lambda'^3}{J} + \mathcal{O}\left(\frac{\lambda'}{J^2}\right) \quad (142)$$

where the last three terms give the finite size corrections which coincide with those computed for this state in sec. 3 and 4, see eq.s (60) and (127). Again we computed the finite-size corrections from the Bethe equations up to the order λ'^8 and we found perfect agreement with the string theory result for

⁶This situation corresponds to the state $|s\rangle$ on the string theory side, so we label the corresponding energy/scaling dimension as E_s .

the energy E_s . We conclude that the dispersion relation up to the first order in $\frac{1}{J}$ for two magnons both in the same $SU(2)$ sector is

$$E_s = 2\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 1 + \frac{\lambda'}{J} \frac{4\pi^2 n^2}{\frac{1}{4} + 2\pi^2 n^2 \lambda'} \left(\sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} - 2\pi^2 n^2 \lambda' \right) \quad (143)$$

in the limit of large λ with $\lambda' = \lambda/J^2$ fixed.

In this section we have thus given evidence that the all loop Bethe equations proposed in [12], with the rapidities, the dressing phase and the charges constructed here for the $SU(2) \times SU(2)$ sector, are consistent with the finite size corrections computed directly from the string sigma model and the corresponding LL model.

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